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Equilibrium and relaxation in turbulent wakes

By R. NARASIMHA† AND A. PRABHU
Department of Aeronautical Engineering, Indian Institute of Science, Bangalore

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In order to study the memory of the larger eddies in turbulent shear flow, experiments have been conducted on plane turbulent wakes undergoing transition from an initial (carefully prepared) equilibrium state to a different final one, as a result of a nearly impulsive pressure gradient. It is shown that under the conditions of the experiments the equations of motion possess self-preserving solutions in the sense of Townsend (1956), but the observed behaviour of the wake is appreciably different when the pressure gradient is not very small, as the flow goes through a slow relaxation process before reaching final equilibrium. Measurements of the Reynolds stresses show that the approach to a new equilibrium state is exponential, with a relaxation length of the order of 10^3 momentum thicknesses. It is suggested that a flow satisfying the conditions required by a self-preservation analysis will exhibit equilibrium only if the relaxation length is small compared with a characteristic streamwise length scale of the flow.

1. Introduction

Relatively few detailed studies have been made of relaxing turbulent shear flows (see Tani (1969) for a recent survey of earlier work), and the unsatisfactory state of our knowledge of the subject has been emphasized by Coles (1969), who called it (after completing an exhaustive examination of all available data) "the darkest corner in the experimental picture". The present work concentrates on relaxing wakes perturbed from equilibrium by a pressure gradient (we hope that the postponement of the definition of the terms to the end of this section will not make the following remarks entirely unintelligible). The classic work of Townsend (1949, 1956) has shown how a study of the plane wake has provided much fundamental understanding of equilibrium turbulent flow; the investigation of *relaxing* wakes is attractive not only because well-defined equilibrium states exist and can be realized experimentally, but also because of the result – to be demonstrated below in § 2 – that self-preserving solutions can be found for arbitrary pressure gradients provided that the wake defect velocity remains sufficiently small. This linearizing assumption, which is quite natural from experience with the constant-pressure wake, does not impose as severe a restriction on the validity of the analysis as it may appear to do at first sight; it is for example entirely adequate, as we shall see, for describing the experiments of Gartshore (1967) on self-preserving wakes in adverse pressure gradients, although his own analysis did not use the assumption.

† Present address: Department of Mathematics, University of Strathclyde, Glasgow.

These linear self-preserving solutions have provided a framework for the present experiments, which were undertaken in the spirit of Clauser's 'black box' (Clauser 1956): a carefully prepared equilibrium wake was subjected to a nearly impulsive pressure gradient by changing the free-stream velocity from its initial value U_1 to a final value U_2 over a relatively short distance. The resulting shock-like transition from one equilibrium state to another is found not to obey the self-preserving solution although the conditions required by the analysis are fulfilled in the experiments; it follows that no local theory (e.g. Prabhu 1966) will be valid either. Instead the flow undergoes a slow relaxation process, the final equilibrium state being attained exponentially to a good first approximation.

Before proceeding further it is necessary to consider what meaning the word equilibrium should carry, for, as the discussions at the Stanford Conference (Kline *et al.* 1969) revealed, there is no universal agreement on this question. Clauser (1954) used the word to denote the existence of similarity in the mean velocity distribution in the outer part of a turbulent boundary layer with the choice of appropriate length and velocity scales. Observation shows that in the wake the defect velocity profiles have the same nearly Gaussian shape, to within experimental error, at any rate, even in quite strong pressure gradients; taken together with the failure of local theories already noted, this suggests that mere similarity in velocity distributions is not always an adequate criterion on which to base a useful equilibrium concept.

Following the idea of self-preservation and moving equilibrium discussed by Townsend (1956), we adopt here the operational definition that: *a turbulent shear flow is in equilibrium in a region if, at every streamwise station in the region, the distributions of mean velocity and the turbulent stresses exhibit similarity with essentially the same scales.* To clarify the terminology, consider as an illustration the two variables w and τ, respectively an appropriate mean velocity (e.g. the wake defect) and Reynolds stress (e.g. shear). Each separately exhibits similarity if we can write

$$w(x,y) = w_0(x) f[y/\delta(x)], \quad \tau(x,y) = \tau_0(x)\, g[y/\delta_\tau(x)], \tag{1.1}$$

where x and y are co-ordinates along and normal to the main stream, and w_0, τ_0, δ and δ_τ are suitable local scales. In *equilibrium,* these scales are inter-related such that†

$$w_0(x)/\tau_0^{\frac{1}{2}}(x), \quad \delta(x)/\delta_\tau(x) \tag{1.2}$$

(and similar ratios of scales of all other relevant mean velocity and stress components) are independent of x. Townsend's self-preservation analysis deduces, from only the mean conservation equations of mass and momentum, the (necessary) conditions for equilibrium to prevail, and furthermore obtains self-preserving solutions for the flow which prescribe functional forms of the scales $w_0(x)$, $\delta(x)$ etc. All equilibrium flows are self-preserving, but the need to make a distinction arises because (to anticipate our chief conclusion) the existence of self-preserving solutions is not sufficient to ensure that the corresponding flow will be in equilibrium. This possibility, which has already been recognized (e.g Townsend 1956, p. 93), here is explicitly related to the relaxation characteristics

† We use kinematic units for the stresses.

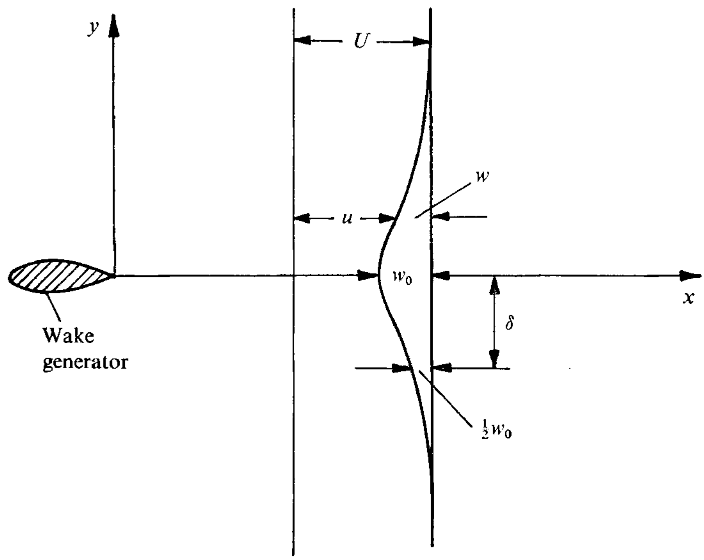

FIGURE 1. Sketch defining notation.

of the shear flow. A second reason for making the distinction is that it seems more appropriate to call a related concept which we shall need later (Prabhu & Narasimha 1972) local equilibrium rather than local self-preservation.

2. The self-preservation analysis

If all viscous and the normal turbulent stresses are neglected, the equations governing the development of a plane incompressible turbulent wake are, in the boundary-layer approximation,

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{2.1}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=U\frac{dU}{dx}+\frac{\partial \tau}{\partial y}. \tag{2.2}$$

The notation is defined in figure 1. Consider now shallow wakes, in which the maximum value of the defect velocity $w(x,y)\equiv U(x)-u(x,y)$, say $w_0(x)$, is much smaller than the free-stream velocity $U(x)$ at each streamwise station x. If $\partial w/\partial x$ can be neglected in (2.1), we have $v\simeq -y\,dU/dx\equiv -yU'$, and (2.2) simplifies to

$$U\frac{\partial w}{\partial x}-U'y\frac{\partial w}{\partial y}+U'w=-\frac{\partial \tau}{\partial y} \tag{2.3}$$

(dropping higher order terms in w_0/U). With τ taken as proportional to w_0^2, this equation shows that the appropriate pressure-gradient parameter is the ratio of longitudinal to shear strain rate:

$$\lambda\equiv U'\delta/w_0, \tag{2.4}$$

where δ is some measure of the wake thickness. The strain ratio can be of order unity without violating the boundary-layer approximation, provided that the length scale associated with streamwise variations of the flow quantities, say U/U', is much larger than δ; i.e. if

$$\lambda w_0/U \ll 1. \tag{2.5}$$

If $U' = 0$ the appropriate length scale is x instead of U/U'. Condition (2.5) will be amply satisfied in the experiments to be reported here.

Following Townsend (1956), we now seek self-preserving solutions by putting

$$w(x,y) = w_0(x)f(\eta), \quad \tau(x,y) = w_0^2 g(\eta), \eta \equiv y/\delta(x); \tag{2.6}$$

from (2.3) we then get the equation

$$\left[\frac{1}{w_0}\frac{d}{dx}(U\delta)\right]\eta f' - \left[\frac{\delta}{w_0^2}\frac{d}{dx}(Uw_0)\right]f = g', \tag{2.7}$$

where primes on f and g denote differentiation with respect to η. To satisfy the requirements of self-preservation the terms in square brackets in (2.7) must be independent of x. This gives us two conditions on the three quantities U, w_0 and δ, with the immediate consequence that for arbitrary $U(x)$ the other two quantities are determined, or equivalently that self-preserving solutions exist for arbitrary pressure gradients.

These solutions can be written down in a particularly simple form in terms of the quantity

$$M \equiv U^2 w_0 \delta, \tag{2.8}$$

which must be independent of x in a shallow wake with similar velocity profiles. This result follows by integrating (2.7) or from the Kármán momentum integral equation, by noting that the shape factor for a shallow wake is approximately unity and that M is proportional to U^3 times the momentum thickness θ. A consequence of the constancy of M is that the coefficients in (2.7) are equal; if their common value is K_1, the wake development is given by

$$\frac{(U\delta)^2}{M^2} = \frac{1}{(Uw_0)^2} = \frac{2K_1}{M}\int\frac{dx}{U(x)} + \text{constant}, \tag{2.9}$$

which implies that, for any $U(x)$, the quantities $U\delta$ and Uw_0 in the self-preserving solutions have the same dependence on the time-of-flight variable

$$T \equiv \int \frac{dx}{U(x)} \tag{2.10}$$

as in a constant-pressure wake. Results similar to (2.9) have been given earlier (Hill 1962; Prabhu 1966), but such derivations have been based, in contrast to the present approach, on some specific model for the turbulence. Of course such models will also provide explicit solutions for the velocity profile (whatever their worth). For later reference, we note that with an eddy viscosity ν_T defined by

$$\tau = -\nu_T\,\partial w/\partial y, \quad \nu_T = k_0 w_0 \delta, \tag{2.11}$$

and k_0 assumed to be a universal constant, the relevant solution of (2.7) can be written as

$$f(\eta) = \exp(-\eta^2 \ln 2) \tag{2.12}$$

with suitable normalization (K_1 in (2.9) now being $2k_0 \ln 2$).

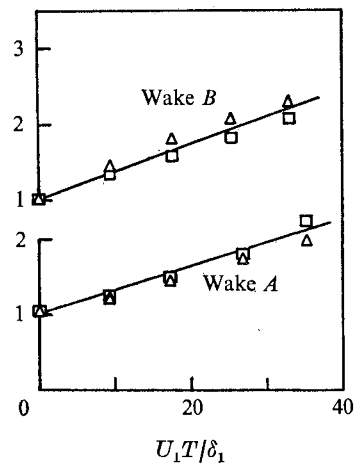

FIGURE 2. Wake development in Gartshore's experiments. $\triangle$, $(U\delta/U_1\delta_1)^2$; $\square$, $(Uw_0/U_1w_{01})^{-2}$; ——, linear self-preservation theory, with $K_1/2\ln 2 = 0{\cdot}065$. T is measured from an initial station x_1, whose position, in terms of the width of the square rod generating the wake, is 72·2 in wake A ($\lambda \simeq 0{\cdot}043$) and 66 in wake B ($\lambda \simeq 0{\cdot}02$).

If, as suggested by Townsend (1956) for a constant-pressure wake, more general solutions contain a term of order $x^{\frac{1}{2}}$ on the right-hand side of (2.9), it is preferable to write the present solutions in the form

$$U^2\delta^2/MT = 2K_1 = M/U^2w_0^2T \tag{2.13}$$

to emphasize their asymptotic nature for large x. When U is constant so is θ (by momentum conservation), and the parameters $\delta^2/x\theta$ and $U^2\theta/w_0^2x$, proportional to those in (2.13), are then more convenient.

To assess the power of the crucial assumption of small defects in the above 'linear' theory, we present a comparison (figure 2) between our theory and two sets of wake measurements reported by Gartshore (1967). In these experiments the defect ratios were maintained approximately constant at 0·19 and 0·24 respectively, through the use of a suitable external pressure gradient.† The excellent agreement shown in figure 2 establishes the useful conclusion that linearization of the inertia terms should be entirely adequate for defect ratios at least as large as a quarter.

3. The experiments

We report here the results of measurements in two flows, designated F 1 and F 2. These were part of a wider series described by Prabhu & Narasimha (1972), where additional details will be found.

† The experiments were in fact accompanied by an analysis, *not* limited to small defects, that showed that self-preserving solutions exist only when w_0/U and δ/x are constant and U is proportional to x^m. When linearization is permissible these conditions implying constant λ are unnecessary, as we have seen, but of course the value of m ($= -\frac{1}{3}$) required for a small (and constant) defect ratio in Gartshore's theory is identical to that deduced from (2.9).

3.1. *Apparatus*

The experiments were conducted in a 1×1 ft tunnel whose test-section length had to be extended to 14 ft from the original 12 ft because experiment showed the greater length to be necessary for attainment of the final equilibrium state. The short favourable pressure gradients imposed on the wake in both F 1 and F 2 were obtained by the use of side-wall liners which produced a contraction in the tunnel cross-sectional area over a distance of about 6 in. The free-stream velocity distributions $U(x)$ in the experiments are shown in figures 5 and 6 below. A sketch of the tunnel will be found in Prabhu & Narasimha (1972).

All turbulence measurements reported here were made with Pt–Rh probes, the wire diameter being 0·0001 in. for straight wires and 0·0002 in. for X-wires. A constant-current hot-wire amplifier and average-square computer built in the laboratory (Prabhu 1968) were used for processing the signals. Some measurements were repeated with a Shapiro–Edwards set; the differences were always less than 3 %. Velocity profile measurements were made with a round Pitot tube of outer diameter 1 mm and a micromanometer reading accurate to within 0·05 mm alcohol.

3.2. *Wake generator*

After trials with various bodies, the twin-plate configuration shown in figure 3 was adopted to generate a wake, as it fulfilled the obvious practical requirement (important for the studies contemplated here) of relatively early attainment of equilibrium. This is demonstrated in figure 3 using earlier measurements of the wake behind a circular cylinder (Townsend 1949; Uberoi & Freymuth 1969). The parameters chosen here for illustration, namely $\delta^2/x\theta$ and $\tilde{u}_c/w_0$ (where $\tilde{u}_c$ is a characteristic root-mean-square value of the longitudinal velocity fluctuations), should become independent of x in equilibrium, as discussed in §§ 1 and 2; they are preferable to the more usual combinations of the form $(\delta/\theta)^2$ or $(U/\tilde{u})^2$, whose plots against x suffer from a basic uncertainty about the virtual origin (Townsend 1956, p. 137).

It is clear from figure 3 that both the manner and the rate of approach to equilibrium can depend strongly on the initial conditions: the slow equilibration in the case of the circular cylinder is presumably due to the violent motion in the near wake following separation on the body. The asymptotic values of $\delta^2/x\theta$ show good agreement, but the best that can be said for $\tilde{u}_c/w_0$ is that the different observations are not inconsistent, as variations are noticeable in the circular-cylinder measurements even for $x/\theta > 10^3$. The attainment of equilibrium in the present measurements is more closely demonstrated by the $\tilde{u}$ and τ distributions across the wake, to be discussed in § 5.

4. Relaxation of mean flow quantities

Although the turbulent energy and stresses may be expected to provide the more fundamental information on the nature of relaxing flows, it is worthwhile to consider first the observed development of the mean flow in relation to the self-preservation theory.

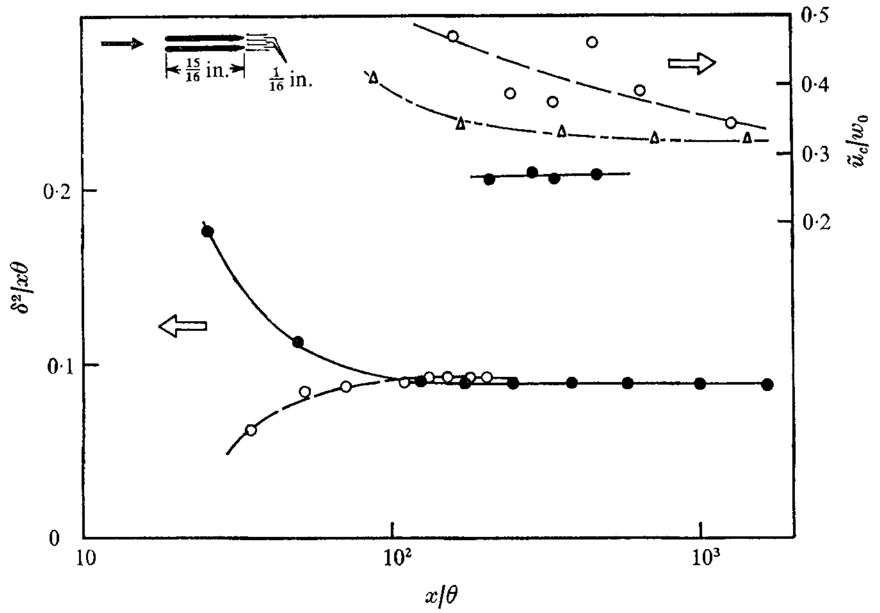

FIGURE 3. Streamwise variation in constant-pressure flow of wake thickness and characteristic turbulence intensity, in the form of parameters which should settle down to constant values in the asymptotic equilibrium state. ○, Townsend (1956, figure 7·1); △, Uberoi & Freymuth (1969); ●, present measurements. For the former two sets of data obtained behind a circular cylinder, θ has been estimated using the standard drag coefficient curves (e.g. Schlichting 1955, figure 1·4); only the shape of the $\tilde{u}_c/w_0$ graph is significant, as the numerical values contain an unknown factor. The wake generator used in present work is shown in the inset; $U = 64{\cdot}6$ ft/s $\tilde{u}_c$ = maximum value of $\tilde{u}$ at each station.

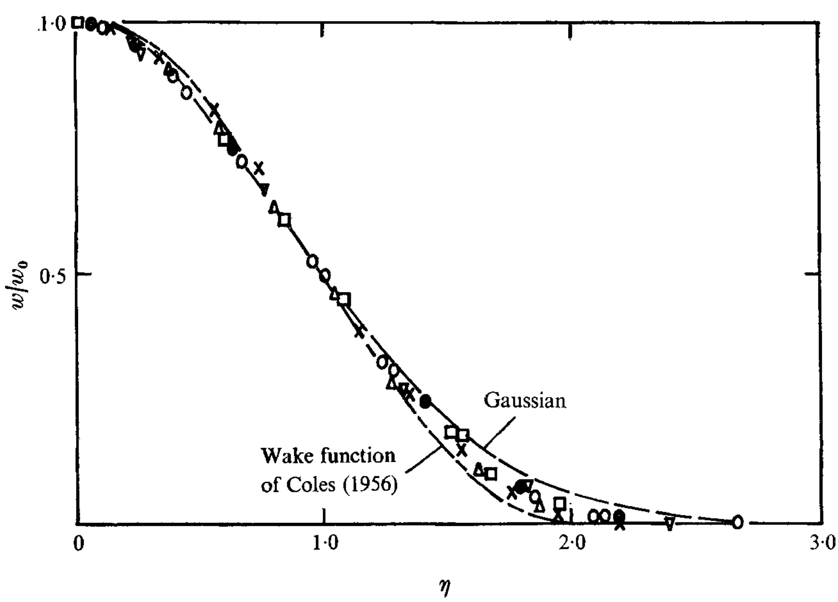

FIGURE 4. Defect velocity profiles selected randomly from the measurements of Prabhu & Narasimha (1972) and Townsend (1949). ○, flow F2 ($\lambda = 0$); ×, F3 ($\lambda = 0{\cdot}35$); ●, M1 ($\lambda = -0{\cdot}152$); □, A1 ($\lambda = -0{\cdot}186$); ▽ F4 ($\lambda = 1{\cdot}57$); △, Townsend ($\lambda = 0$).

4.1. *Mean velocity profiles*

Figure 4 shows, in the normalized variables w/w_0 and η, several defect velocity profiles chosen randomly from the various experiments reported by Prabhu & Narasimha (1972), but including flows not in equilibrium in the sense of our definition in § 1. It is perhaps not very surprising that pressure gradients and departure from equilibrium have no significant effect on the normalized profile; for all practical purposes the experimental data define a unique function, say

$$w/w_0 = \bar{f}(\eta). \tag{4.1}$$

(A table of this function and some related integrals are given by Narasimha & Prabhu (1971).) Also shown in figure 4 are the Gaussian curve (2.12) and the wake function of Coles (1956), suitably renormalized so that $\bar{f}(0) = 1, \bar{f}(1) = \frac{1}{2}$. The approach to the free-stream velocity is faster than that of the Gaussian curve, as in the constant-pressure wake (Townsend 1956), but slower than that of Coles's wake function.

4.2. *Mean flow parameters*

Data for δ and w_0 from experiments F 1 and F 2 are shown in figures 5 and 6, respectively, in non-dimensional variables in which the linear self-preserving solution plots as a straight line. Also shown in the diagrams are the free-stream velocity distributions $U(x)$, the strain ratio λ and the quantity M/M_1, where M_1 is the value of M (defined by (2.8)) at some suitable initial station. If the two-dimensional momentum integral is obeyed $M/M_1 = 1$ for all x, so that observed departures of M/M_1 from unity provide a measure of the lack of two-dimensionality in the flow.

It is clear that even with the small strain ratio of F 1 there is no complete agreement with the self-preserving solution downstream of the pressure gradient, but the differences (revealed only because measurements were made at a large number of streamwise stations) are not large; they can if one wishes be attributed, over the range of the measurements, to a shift in the virtual origin, which is, however, an ambiguous and limited concept (as discussed in § 2). Even this unsatisfactory way of describing the effect of the pressure gradient is not available when the strain ratio is higher, as in F 2: here no shift in origin will account for discrepancies in both δ and w_0. The slight departures in this flow from strict two-dimensional momentum balance far downstream can be allowed for by making convergence corrections as described by Prabhu & Narasimha (1972), but these are too small to account for the observed differences. The adequacy of the linear approximation in the theory cannot be questioned either, because its validity for much higher defect ratios than in F 2 has been demonstrated in § 2.

We must therefore look to the strain ratio as a possible parameter governing the applicability of the self-preserving solutions; from an examination of its values in the present experiments as well as in Gartshore's (in which λ is respectively positive and negative), we may tentatively conclude that the absolute value of λ must be sufficiently small in some sense for self-preserving solutions to be observed.

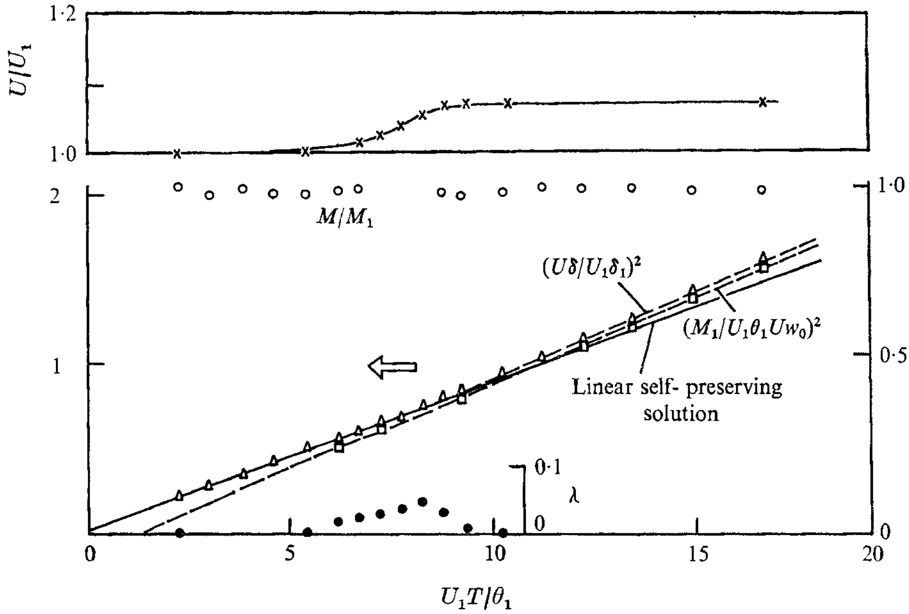

FIGURE 5. Wake development in flow F1. The self-preserving solution corresponds to $K_1/2\ln 2 = 0{\cdot}065$. Dashed lines through the experimental points produce backwards to a slightly shifted virtual origin. Points for w_0 are not shown separately where they are indistinguishable from those for δ.

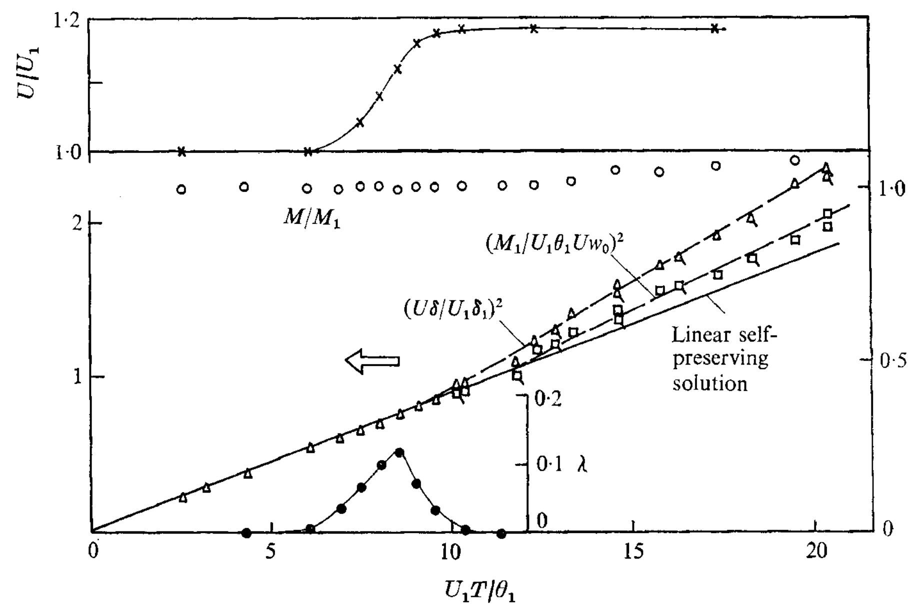

FIGURE 6. Wake development in flow F2. Same K_1 as in F1. For $U_1T/\theta_1 < 10$, points for w_0 are indistinguishable from those for δ.

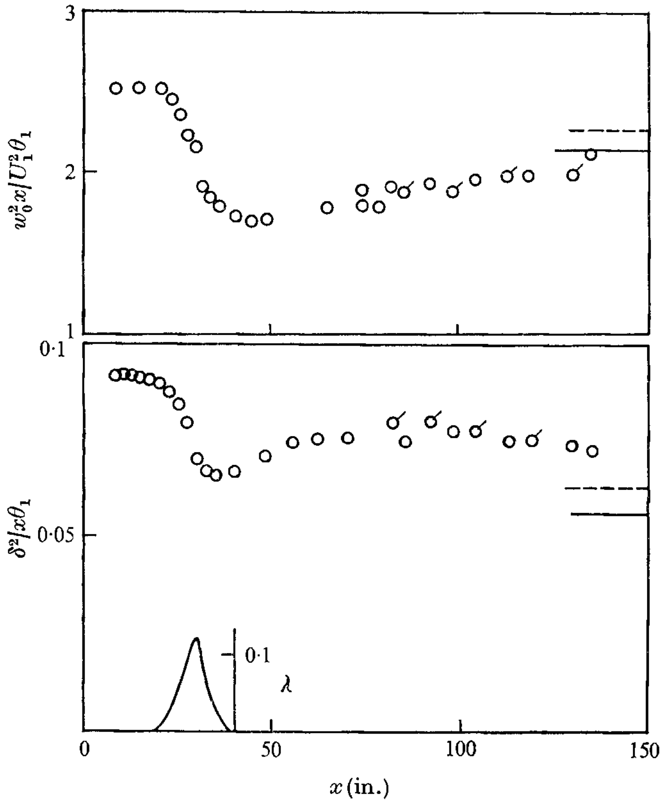

FIGURE 7. The approach to new equilibrium values of wake defect and thickness parameters in F2. Final equilibrium: ——, uncorrected values; – – –, 'corrected' values allowing for the effective convergence of the flow. Flagged points are repeats.

That the 'discrepancies' noted in figure 6 might in fact be described as relaxation of the flow is shown graphically in figure 7, where we examine the variation of parameters of a type suggested in § 2, namely $\delta^2/x\theta_1$ and $w_0^2x/U_1^2\theta_1$; here and in what follows subscript 1 refers to the initial equilibrium state, before the application of the pressure gradient, and 2 to the final asymptotic equilibrium state downstream of the pressure gradient. Using bars to denote equilibrium values, we must have, asymptotically as $x \to \infty$,

$$\left.\frac{\delta^2}{x\theta_1}\right|_2 \approx \left[\frac{\overline{\delta^2}}{x\theta}\right]\frac{\theta_2}{\theta_1}, \quad \left.\frac{w_0^2 x}{U_1^2\theta_1}\right|_2 \approx \left[\frac{\overline{w_0^2 x}}{U^2\theta}\right]\frac{U_2^2\theta_2}{U_1^2\theta_1}, \tag{4.2}$$

where the quantities in square brackets are universal numbers and are listed in the appendix (along with some others) for convenient reference. The expected final equilibrium value of each variable, calculated using (4.2), with and without convergence corrections, is shown in figure 7. It is obvious that the flow in F 2 is an extremely slow relaxation from one equilibrium state to another.

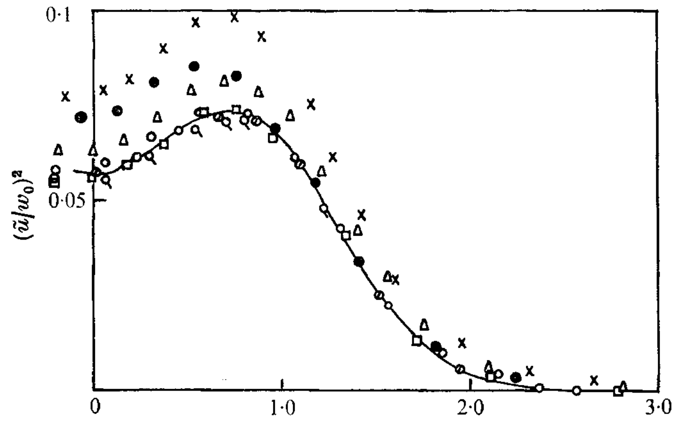

FIGURE 8. Distribution of longitudinal turbulence intensity across the wake in F 2, using mean flow scales. ——, equilibrium; ○, $U_1T/\theta_1 = 3{\cdot}35$ ($\lambda = 0$); ⊘, 4·60 ($\lambda = 0$); □, 5·8 ($\lambda = 0$); △, 8·35 ($\lambda = 0{\cdot}12$); ×, 9·70 ($\lambda = 0{\cdot}02$); ●, 21·4 ($\lambda = 0$). Flagged points are repeats.

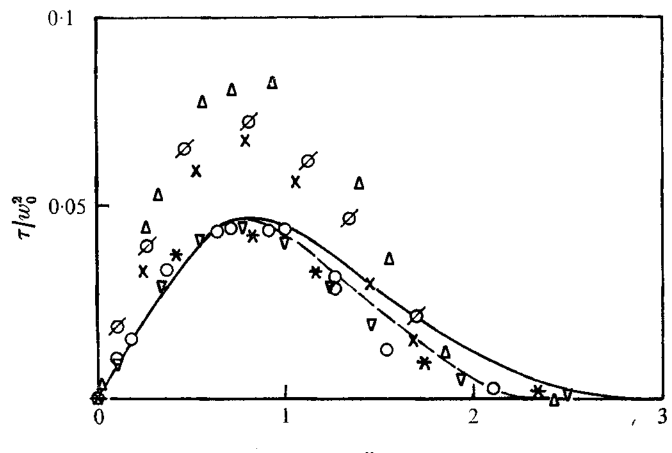

FIGURE 9. Reynolds shear stress distributions across the wake in F2. ○, $x = 10$ in.; ▽, 15 in.; $*$, 20 in.; ⊘, 35 in.; △, 44 in.; ×, 53 in. Equilibrium distribution: ——, Gaussian velocity profile; – – –, true measured velocity profile. λ is 0·025 at $x = 35$ in. and zero at the other four stations.

5. Relaxation in the Reynolds stresses

To obtain a more direct and quantitative indication of the approach to equilibrium fairly extensive measurements of the Reynolds stresses (both normal and shear) were made in F 2. Figures 8 and 9 show the distributions of $\tilde{u}/w_0$ and τ/w_0^2 across the wake at various stations. Upstream of the pressure gradients each of these quantities is seen to define a unique function of η, thus confirming the existence of equilibrium. On the other hand the strong departures which are

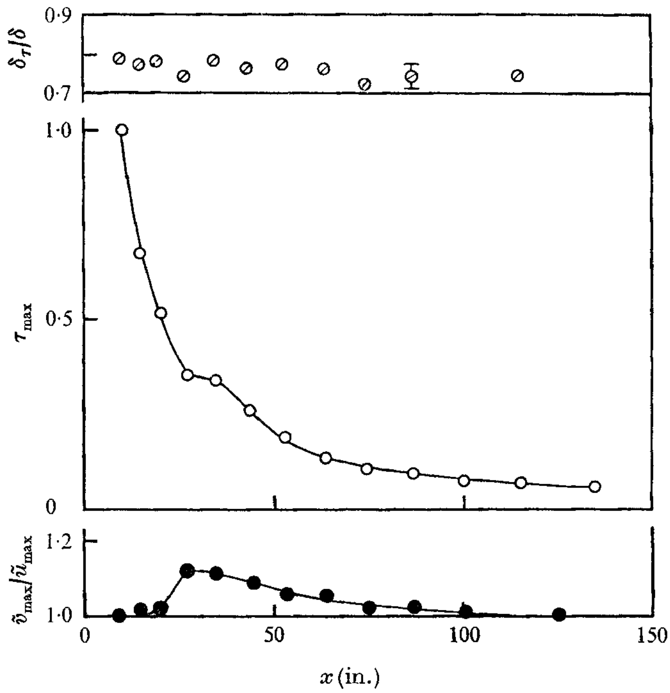

FIGURE 10. Streamwise distribution of characteristics of the Reynolds stress distributions in F 2. Both τ_{max} and $\tilde{v}_{max}/\tilde{u}_{max}$ are normalized by their values at the first station. δ_τ is defined as the value of y where τ is maximum, and the bar on the curve shows the uncertainty in its determination.

noticeable even far downstream of the pressure gradient are sure evidence of a slow relaxation.

If one insists on using an eddy viscosity, then τ/w_0^2 is easily seen to be proportional to the coefficient k_0 of (2.11). The large variations in τ/w_0^2 revealed by figure 9 therefore imply correspondingly large variations in the eddy viscosity, which does not augur well for at least those simpler theories assuming k_0 to be a universal constant. Interestingly enough, the shear stress itself does not show strong variations; examination of a characteristic value at each station, illustrated in figure 10, provides unmistakable evidence that during the straining caused by the pressure gradient the stress hardly changes. We conjecture that this 'stress freezing' is a phenomenon characteristic of rapid distortion, of the type discussed by Batchelor & Proudman (1954) for homogeneous and (initially) isotropic turbulence. The conjecture rests at present on two grounds. First, it explains the observed increase in the ratio of the normal to longitudinal fluctuating velocities, $\tilde{v}/\tilde{u}$ (figure 10); second, the relevance of rapid distortion effects to highly accelerated turbulent shear flows has been demonstrated in recent work on a contracting two-dimensional channel flow (Ramjee, Badri Narayanan & Narasimha 1972). It is at least plausible that an increase in $\tilde{v}$ relative to $\tilde{u}$ would counteract the decrease in shear stress that would have occurred had the flow been in equilibrium through the pressure gradient. However, a satisfactory theoretical treatment of rapid distortion of shear flow turbulence is necessary before these arguments can be tested in detail.

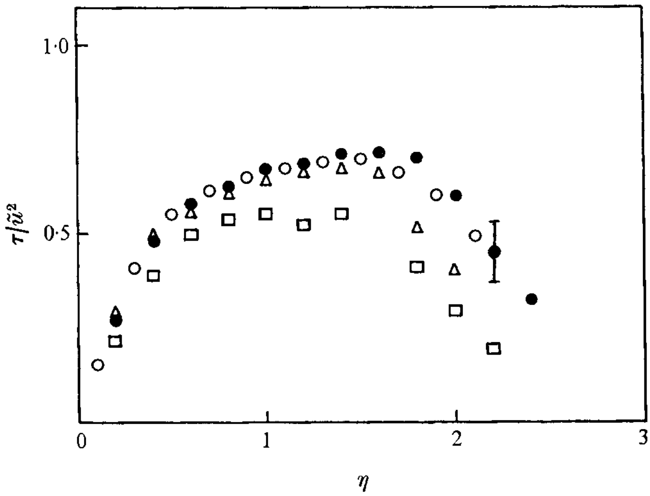

FIGURE 11. Distribution of $\tau/\tilde{u}^2$ across the wake in F2. ●, $x \simeq 10$ in.; ○, 15 in.; △, 35 in.; □, 87 in.; bar shows probable error in evaluating coefficient at a typical point in the tail.

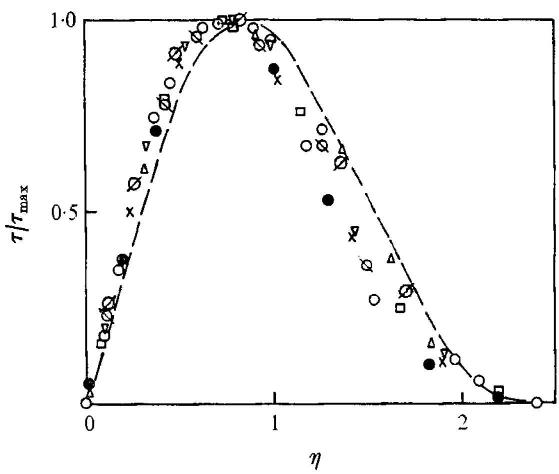

FIGURE 12. Internal similarity in Reynolds shear stress distribution across wake in F2, using τ_{max} and δ as scales. ○, $x = 10$ in.; ▽, 15 in.; □, 27 in.; ⌀, 35 in.; △, 44 in.; ×, 53 in.; ●, 87 in.; ⧄, 135 in. ---, stress distribution calculated from the measured velocity profile making the same assumptions as in linear self-preservation theory.

Although the scales characterizing the mean velocity and Reynolds stress are thus essentially different in non-equilibrium, certain kinds of internal similarity still persist. Figure 11 shows that $\tau/\tilde{u}^2$ varies much less during relaxation than τ/w_0^2, thus lending support to the kind of hypothesis made by Bradshaw *et al.* (1967) in their method of computing turbulent boundary layers. A more obvious kind of similarity is that shown in figure 12, where local scales, as in (1.1), have been adopted: the *shape* of the stress distribution curve, like that of the velocity distribution, is hardly affected by non-equilibrium conditions. What is interesting (and potentially useful for calculations of wake development) is that the length

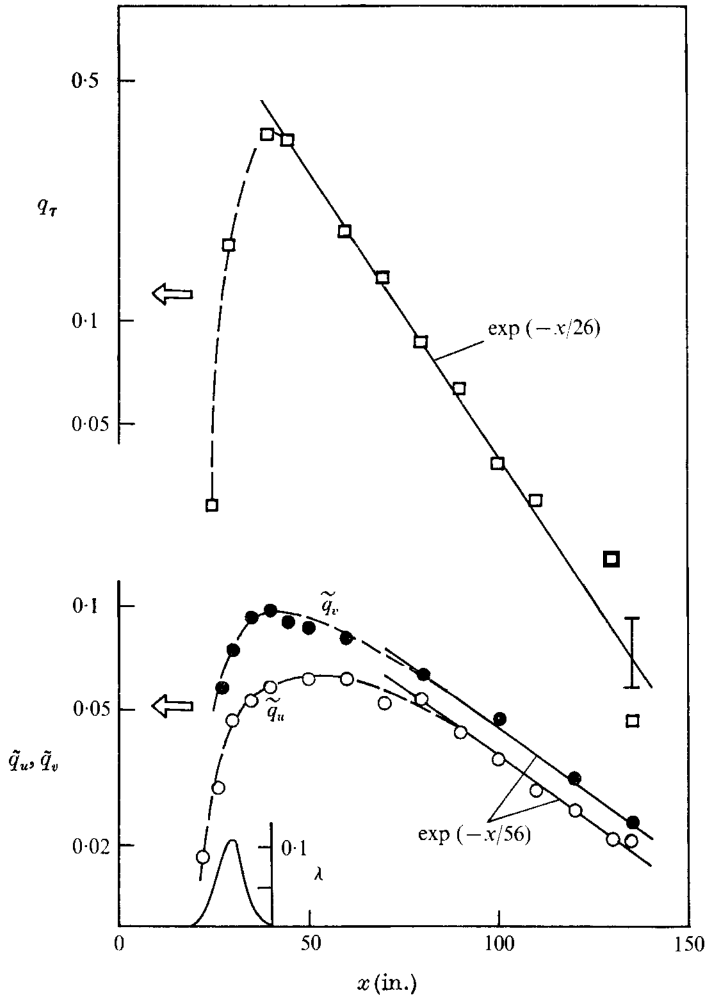

FIGURE 13. Approach to new equilibrium in F2 as shown by three measures of departure from equilibrium. Bar on the q_τ curve corresponds to a 1 % error in measurement of τ.

scales $\delta(x)$ and $\delta_\tau(x)$ keep in step with each other, so that the ratio $\delta(x)/\delta_\tau(x)$ of (1.2) is independent of x (figure 10) even when $w_0(x)/\tau_0^{\frac{1}{2}}(x)$ is not. The normal stresses $\tilde{u}^2$ and $\tilde{v}^2$ are also found to exhibit the same kind of similarity, but the data will not be presented here.

This special kind of internal similarity enables us to devise simple measures of the departure from equilibrium at any given streamwise station. Figure 13 shows the variation with x of three such measures, defined by

$$\left.\begin{aligned} \tilde{q}_u \equiv \frac{\tilde{u}_{\max}}{w_0} - \frac{\bar{\tilde{u}}_{\max}}{\bar{w}_0}, \quad \tilde{q}_v \equiv \frac{\tilde{v}_{\max}}{w_0} - \frac{\bar{\tilde{v}}_{\max}}{\bar{w}_0}, \\ q_\tau \equiv \frac{\tau_{\max}}{w_0^2} - \frac{\bar{\tau}_{\max}}{\bar{w}_0^2}, \end{aligned}\right\} \tag{5.1}$$

Experiment	U_2 (ft/s)	θ_2 (in.)	$\tilde{L}_u$ (in.)	$10^4 \times (\theta_2/\tilde{L}_u)$
F 2a	43·9	0·0236	63	3·8
F 2b = F 2	76·2	0·0243	56	4·3
F 2c	95·2	0·0239	58	4·1

TABLE 1. Measured relaxation lengths

where bars denote equilibrium values (listed in the appendix, as determined from present measurements at constant pressure). It is seen that at least to a good first approximation the approach to equilibrium in the final stages is exponential. The data for q_τ are the most convincing in this respect, although the last couple of points show appreciable scatter, chiefly because there q_τ involves small differences between nearly equal quantities. In the region where the behaviour is exponential we can define a relaxation length L such that $q \sim \exp(-x/L)$. It is seen that the observed values of L are generally quite large; $\tilde{L}_u$ and $\tilde{L}_v$ (which are practically the same) are particularly so. Even L_τ, which is only about half of $\tilde{L}_u$, is of the order of 2 ft. The departure from equilibrium is sufficient to be noticeable as far downstream as 8 ft from the pressure gradient perturbation!

It is reasonable to suppose that it should be possible to describe the final stage of relaxation as a perturbation on the new equilibrium state, and hence that the relaxation parameters are determined by the asymptotic values of the wake variables. As an equilibrium wake is, however, completely characterized by U and θ, the argument suggests that L must be proportional to θ_2 and (on dimensional grounds) independent of U_2. This has been confirmed by experiment, by making relaxation measurements of $\tilde{u}$ at three different free-stream velocities with the same experimental configuration as in flow F 2. The results are shown in table 1. More data would obviously be desirable, but we may tentatively conclude that $\theta/L \simeq 4 \times 10^{-4}$.

6. Discussion

From the experiments described here, we find that, while a wake may reach equilibrium more quickly behind particularly favourable wake-generating configurations, a perturbation on this equilibrium wake in the form of an impulsive pressure gradient results in a decidedly slow relaxation process towards the new equilibrium state. Within the pressure gradient region, the turbulence appears to respond as to a sudden distortion, at least qualitatively. It is interesting that, while the mean velocity and Reynolds stress scales quickly depart from their characteristic equilibrium inter-relation on the imposition of the pressure gradient, the length scales describing the distribution of the same quantities across the flow nearly maintain their equilibrium ratio.

The measurements indicate that we may take the final approach to equilibrium to be exponential, at least as a working hypothesis. The rather large relaxation lengths so defined are consistent with the marked effect of initial conditions on the growth of a mixing layer as observed by Bradshaw (1966).

The chief conclusion we draw from the experiments is that the existence of self-preserving solutions of the mass and momentum equations, obtained by the usual analysis, is no guarantee that the corresponding flows will actually be observed, even though the conditions required by the analysis are apparently fulfilled. Clearly, additional conditions must be satisfied in order that equilibrium may prevail, and it is natural to expect that they will involve the relaxation length L introduced in § 5. It should thus be necessary that L be much smaller than a length characteristic of the pressure gradients, say U/U'. Recalling that $L \sim 10^3\theta$ and $U\theta = O(w_0\delta)$, the condition can be written as

$$\lambda(w_0/U)^2\, 10^3 \ll 1. \tag{6.1}$$

For $(w_0/U)^2 \sim 10^{-3}$, which is typical of the measurements reported here, the above condition is consistent with the requirement that λ be small for self-preserving solutions to provide an adequate representation of the flow.

It has of course been long realized (Townsend 1956) that such a self-preserving solution, even when it exists, can in general be expected to hold only 'eventually'. However, the explicit recognition of a chracteristic relaxation time for a turbulent shear flow implies that a large number of possible solutions can never be attained in reality, irrespective of how long the flow is permitted to develop.

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Appendix. Equilibrium wake parameters

Some characteristic parameters for equilibrium wakes, as found from present measurements, are listed below.

$$\overline{\delta}^2/x\theta = 0{\cdot}089, \quad \overline{w}_0^2 x/U^2\theta = 2{\cdot}34,$$

$$\overline{\tau}_{\max}/\overline{w}_0^2 = 0{\cdot}045, \quad \overline{\tilde{u}}_{\max}/\overline{w}_0 = 0{\cdot}27,$$

$$\overline{\tilde{u}}/\overline{w}_0 = 0{\cdot}24, \quad \overline{\tilde{v}}/\overline{\tilde{u}} = 1{\cdot}18, \quad \text{on the centre-line.}$$

$$\int_{-\infty}^{+\infty} \bar{f}\, d\eta = 2{\cdot}05, \quad \int_{-\infty}^{+\infty} \bar{f}^2\, d\eta = 1{\cdot}50,$$

$$k_0 = 0{\cdot}065, \quad w_0\delta/\nu_T = 1/k_0 = 15{\cdot}4, \quad \theta/L_\tau = 8{\cdot}2 \times 10^{-4}.$$

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